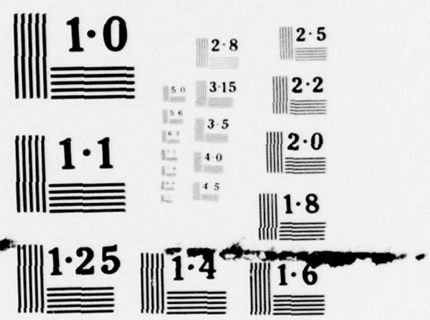
AD-A047 095 CALIFORNIA UNIV BERKELEY DEPT OF MECHANICAL ENGINEERING F/G 20/11
SHELL THEORY FROM THE STANDPOINT OF FINITE ELASTICITY.(U)
SEP 77 P M NAGHDI N00014-75-C-0148

UNCLASSIFIED UCB/AM-77-7 NL

1 OF 1
ADA
047095

END
DATE
FILMED
1-78
DDC



NATIONAL BUREAU OF STANDARDS
MICROCOPY RESOLUTION TEST CHART

OFFICE OF NAVAL RESEARCH

Contract N00014-75-C-0148

Project NR 064-436

Report No. UCB/AM-77-7

Shell Theory from the Standpoint of Finite Elasticity

by

P. M. Naghdi

September 1977

Department of Mechanical Engineering
University of California
Berkeley, California

SHELL THEORY FROM THE STANDPOINT OF FINITE ELASTICITY

P. M. NAGHDI
UNIVERSITY OF CALIFORNIA, BERKELEY

ABSTRACT

This is an account of the nonlinear theory of thin shells from the standpoint of finite elasticity, together with a brief summary of some related recent researches on the subject. The development of the basic theory, as well as a discussion of constitutive equations for elastic shells, are presented via a direct approach on the basis of a continuum model known as a Cosserat surface rather than from the three-dimensional equations of nonlinear elasticity.

1. INTRODUCTION. GENERAL BACKGROUND.

The development of a complete two-dimensional theory of thin elastic shells from the three-dimensional equations of classical continuum mechanics is, in general, a difficult problem. Most of the difficulties arise in the development of the constitutive equations and remain even when the deformation is small. The nature of these difficulties in a derivation from three-dimensional equations has been elaborated upon previously by Naghdi [1, Section 1]. Because of the difficulties just referred to, when dealing with motions and deformations of thin shell-like bodies, it has been customary to employ a variety of approximate procedures in the derivation of elastic shell theories from the three-dimensional equations. Instead of adopting such a procedure, we approach the subject here from another point of view, namely via the theory of a Cosserat (or a directed) surface, which is based on a continuum model comprising a material surface in Euclidean 3-space with a deformable vector field -- called a director -- attached to every point of the surface. It should be emphasized that a Cosserat surface is not just a two-dimensional surface but one which is endowed with structure.

A general background and motivation for use of a direct approach in shell theory, as well as a further description of a Cosserat surface together with some historical background on the subject can be found in [1].[1] Here, we limit ourselves to only a brief historical sketch on the basic theory of Cosserat surfaces. The concept of 'directed' or 'oriented' media originated in the work of Duhem [3] and a first systematic development of theories of oriented media in one, two and three-dimensions (the first two being motivated by rods and shells) was carried

[1] Also, reference may be made to Naghdi [2] which contains a rapid exposition of the theory of a Cosserat surface and includes a discussion of its relevance and applicability to elastic shells and fluid sheets.

out by E. and F. Cosserat [4]. In their work, the Cosserats represented the orientation of each point of their continuum by a set of mutually perpendicular rigid vectors. The purely kinematical aspects of oriented bodies characterized by ordinary displacement and the independent deformation of N deformable vectors in N-dimensional space has been discussed by Ericksen and Truesdell [5], who also introduced the terminology of directors. A complete general theory of a Cosserat surface with a single deformable director given by Green, Naghdi and Wainwright [6] was developed within the framework of thermomechanics; and their derivation in [6] is carried out mainly from an appropriate energy equation, together with invariance requirements under superposed rigid body motions. A related development utilizing three directors at each point of the surface, in the context of a purely mechanical theory and with the use of a virtual work principle, is given by Cohen and DeSilva [7]. A further development of the basic theory of a Cosserat surface is given by Naghdi [1], which also contains additional historical remarks relevant to oriented continua and the development of the theory of elastic shells.

Before describing the contents of this lecture, it is helpful to provide a definition for a shell-like body. To this end, consider a three-dimensional body embedded in a Euclidean 3-space and let its particles be identified by a convected coordinate system[2] θ^i, $(i = 1,2,3)$. Let

$$\underset{\sim}{p} = \underset{\sim}{p}(\theta^1,\theta^2,\theta^3,t) \tag{1.1}$$

denote the position vector, relative to a fixed origin, of a typical particle of the three-dimensional body in the present configuration at time t. In a reference configuration of the body, not necessarily the initial configuration, we denote the position vector by

$$\underset{\sim}{P} = \underset{\sim}{P}(\theta^1,\theta^2,\theta^3) \quad . \tag{1.2}$$

If the reference configuration of the three-dimensional body is specified by the initial configuration, say at time $t = 0$, then the right-hand side of (1.2) can be identified with $\underset{\sim}{p}(\theta^i,0)$. Now, for convenience, set $\theta^3 = \xi$ and adopt the notation

$$\theta^i = (\theta^\alpha,\xi) \quad . \tag{1.3}$$

Keeping the above in mind, we begin by defining first in descriptive terms what is meant by a thin shell or a shell-like body in its reference configuration. Consider a two-dimensional surface, called a reference surface, which may be defined by the parametric equation $\xi = 0$; and let $\underset{\sim}{R}$ and $\underset{\sim}{A}_3$ denote, respectively, the position vector and the unit normal to the reference surface. Imagine now material filaments from above and below surrounding the surface along the normal at each point of the reference surface. Suppose further that the bounding surfaces formed by the end points of the material filaments are equidistant from the reference surface. Such a three-dimensional body is called a shell if the dimension of the body along the normals, called the thickness and denoted by h, is small. A shell is said to be thin if its thickness is much smaller than a certain characteristic length of the reference surface such as the minimum radius of curvature of the reference surface.[3] If h is constant, the shell is said to be of uniform thickness, otherwise of variable thickness. A three-dimensional shell-like body in its reference configuration may be depicted as in Fig. 1, where $\xi = 0$ is shown as the middle surface and the bounding surfaces are specified by $\xi = \pm h/2$ with h a constant. Let the region of space occupied by the shell in the reference configuration be covered by a normal coordinate system (θ^α,ζ) with ζ along the normal to the middle surface and θ^α on the reference surface $(\xi = \zeta = 0)$. Then, the position $\underset{\sim}{P}$ of any point of the shell in

---

[2] Recall that when the particles of a continuum are referred to a convected coordinate system, the numerical values of the coordinates associated with each particle remain the same for all time.

[3] In the case of a plate, since the reference surface is a plane, the characteristic length is taken to be the smallest dimension of the reference plane.

its reference configuration may be expressed as a function of[4] (θ^α,ζ):

$$\underset{\sim}{P} = \underset{\sim}{P}'(\theta^\alpha,\zeta) = \underset{\sim}{R}(\theta^\alpha) + \zeta \underset{\sim}{A}_3(\theta^\alpha) \quad . \tag{1.4}$$

With the above description in mind we now proceed to provide a more general definition for shell-like bodies in fairly precise terms. Since a material surface in the three-dimensional body can be defined by the equation $\xi = \xi(\theta^\alpha)$, it follows that the equations resulting from (1.1) and (1.2) with $\xi = \xi(\theta^\alpha)$ represent the parametric forms of the material surface in the present and the reference configurations, respectively. In particular, the equation $\xi = 0$ defines a surface in space at time t, which we assume to be smooth and nonintersecting. Any point of this surface is specified by the position vector $\underset{\sim}{r}$, relative to the same fixed origin to which $\underset{\sim}{p}$ is referred, where

$$\underset{\sim}{r} = \underset{\sim}{r}(\theta^1,\theta^2) = \underset{\sim}{p}(\theta^1,\theta^2,0,t) \quad . \tag{1.5}$$

Let the boundary of the three-dimensional continuum be specified by the material surfaces

$$\xi = \xi_2(\theta^1,\theta^2) \quad , \quad \xi = \xi_1(\theta^1,\theta^2) \quad , \quad \xi_2 < \xi_1 \quad , \tag{1.6}$$

with the surface $\xi = 0$ lying entirely between them, and a material surface

$$f(\theta^1,\theta^2) = 0 \quad , \tag{1.7}$$

which is chosen such that ξ = const. form closed smooth curves on the surface (1.7).

By way of additional background information, suppose now that $\underset{\sim}{p}$ in (1.1) is a continuous function of θ^i,t, and has continuous space derivatives of order 1 and continuous time derivatives of order 2 in the bounded region $\xi_2 \leq \xi \leq \xi_1$. Hence, to any required degree of approximation, $\underset{\sim}{p}$ may be represented as a polynomial in ξ with coefficients which are continuous functions of θ^α,t. However, instead of considering a general representation of this kind, we restrict attention here to the approximation

$$\underset{\sim}{p} = \underset{\sim}{r} + \xi \underset{\sim}{d} \quad , \tag{1.8}$$

where $\underset{\sim}{r}$ is defined by (1.5) and $\underset{\sim}{d} = \underset{\sim}{d}(\theta^\alpha,t)$. The expression corresponding to (1.8) in the reference configuration is

$$\underset{\sim}{P} = \underset{\sim}{R} + \xi \underset{\sim}{D} \quad , \tag{1.9}$$

where $\underset{\sim}{R} = \underset{\sim}{R}(\theta^\alpha)$ is the reference value of (1.5) and $\underset{\sim}{D} = \underset{\sim}{D}(\theta^\alpha)$ is the reference value of $\underset{\sim}{d}$. Since a representation of the form (1.4) can always be chosen in any one configuration without loss in generality, it is of interest to indicate the relationship between the right-hand sides of (1.9) and (1.4), let $\underset{\sim}{D}$ be specified along the normal to the surface $\xi = 0$, i.e., let

$$\underset{\sim}{D} = D\underset{\sim}{A}_3 \quad , \tag{1.10}$$

where D is the magnitude of $\underset{\sim}{D}$. Then, after equating (1.4) and (1.9), we obtain

$$\zeta = \xi D \tag{1.11}$$

as the transformation relation between ζ and ξ. Hence for a shell-like body, while (1.8) represents an approximation for the position vector in the current configuration at time t, the simple representation (1.9) in the reference configuration with $\underset{\sim}{D}$ specified in the form (1.10) involves no loss in generality.

In the rest of the paper, we consider a fairly general development of the theory of shells by direct approach suitable for application to shell-like bodies as defined in the preceding two paragraphs. Specifically, the next two

---

[4] For details see [1, pages 442 and 471].

sections deal, respectively, with kinematics and the basic principles for shells based on a Cosserat surface and section 4 is concerned with a discussion of constitutive equations for elastic shells. The final section of the paper contains a brief summary of some relevant researches pertaining to elastic shells. An effort is made to employ a direct (coordinate free) notation in keeping with recent trends in finite elasticity and the subject of this symposium. Otherwise our notation, with some minor differences, is essentially that in [1].[5] Throughout the paper, Latin indices (subscripts or superscripts) have the range 1,2,3, Greek indices have the range 1,2 and the usual summation convention is employed.

2. KINEMATICS

Deformable media which are modelled by a material surface, embedded in a Euclidean 3-space, together with K (K = 1,2,...,N) deformable vector fields -- called directors -- attached to every point of the material surface are called Cosserat surfaces or directed surfaces and may be conveniently referred to as $\mathcal{C}_K$. In the absence of the directors, we merely have a two-dimensional material surface which can serve as a model for the construction by direct approach of the membrane theory of shells. With K = 1, the directed medium is a body $\mathcal{C}_1 = \mathcal{C}$ consisting of a material surface and a single deformable director attached to every point of the material surface of $\mathcal{C}$. The latter is the simplest model for the construction of a general bending theory of thin shells; and, for simplicity, we restrict attention to this particular model in the discussion that follows.[6]

Let the particles of the material surface of $\mathcal{C}$ be identified by a system of convected coordinates θ^α ($\alpha = 1,2$); let $\mathcal{s}$, the material surface of $\mathcal{C}$ in the present configuration at time t, be described by its position vector $\underset{\sim}{r}$ relative to a fixed origin; let $\underset{\sim}{a}_\alpha$ and $\underset{\sim}{a}_3$ denote, respectively, the base vectors along the θ^α-curves on $\mathcal{s}$ and the unit normal to $\mathcal{s}$; and let $\underset{\sim}{d}$ stand for the director at $\underset{\sim}{r}$. Then, a motion of the Cosserat surface is defined by vector-valued functions which assign position $\underset{\sim}{r}$ and director $\underset{\sim}{d}$ to each particle of $\mathcal{C}$ at each instant of time, i.e.,[7]

$$\underset{\sim}{r} = \underset{\sim}{r}(\theta^\alpha,t) \quad , \quad \underset{\sim}{d} = \underset{\sim}{d}(\theta^\alpha,t) \quad , \quad [\underset{\sim}{a}_1\underset{\sim}{a}_2\underset{\sim}{d}] \neq 0 \quad , \tag{2.1}$$

where

$$\underset{\sim}{a}_\alpha = \underset{\sim}{a}_\alpha(\theta^\gamma,t) = \frac{\partial\underset{\sim}{r}}{\partial\theta^\alpha} \tag{2.2}$$

and the condition $(2.1)_3$ ensures that the director $\underset{\sim}{d}$ is nowhere tangent to $\mathcal{s}$.

It is convenient to introduce here a slightly different notation than that adopted in Naghdi [1] and a number of previous papers. Thus, we put

$$\underset{\sim}{d}_\alpha = \underset{\sim}{a}_\alpha \quad , \quad \underset{\sim}{d}_3 = \underset{\sim}{d} \tag{2.3}$$

and observe that, in view of $(2.1)_3$ and (2.3), $\underset{\sim}{d}_1,\underset{\sim}{d}_2,\underset{\sim}{d}_3$ are linearly independent vectors. Hence, we may introduce a set of reciprocal vectors $\underset{\sim}{d}^i$ such that

$$\underset{\sim}{d}_i \cdot \underset{\sim}{d}^j = \delta_i^j \tag{2.4}$$

where δ_i^j is the Kronecker symbol in three-space. Whenever desirable, the notations $\underset{\sim}{d}_i = (\underset{\sim}{d}_1,\underset{\sim}{d}_2,\underset{\sim}{d}_3)$ and $(\underset{\sim}{a}_\alpha,\underset{\sim}{d})$ will be used interchangeably throughout the paper depending on the particular context.

Consider now a reference configuration, not necessarily the initial

---

[5] The coordinate free notation employed here is similar to that used by Carroll and Naghdi [8] but some of the definitions are different.

[6] For a more general development of the theory of Cosserat surfaces see for example section 2 of a recent paper by Green and Naghdi [9] which, however, deals with fluid sheets and water waves.

[7] For convenience, we adopt the notation for $\underset{\sim}{r}$ in (1.5) and (1.8) also for the surface $(2.1)_1$. This permits an easy identification of the two surfaces, if desired.

configuration, of the Cosserat surface $\mathcal{C}$. In the reference configuration, let the material surface of $\mathcal{C}$ be referred to by $\mathcal{S}$ with $\mathbf{R}$ as its position vector; let $\mathbf{A}_\alpha, \mathbf{A}_3$ denote, respectively, the base vectors along the θ^α-curves on $\mathcal{S}$ and the unit normal to $\mathcal{S}$; and let $\mathbf{D}$ be the director at $\mathbf{R}$. Then corresponding to (2.1), we have

$$\mathbf{R} = \mathbf{R}(\theta^\alpha) \quad , \quad \mathbf{D} = \mathbf{D}(\theta^\alpha) \quad , \quad [\mathbf{A}_1 \mathbf{A}_2 \mathbf{D}] \neq 0 \quad , \tag{2.5}$$

where

$$\mathbf{A}_\alpha = \mathbf{A}_\alpha(\theta^\gamma) = \frac{\partial \mathbf{R}}{\partial \theta^\alpha} \tag{2.6}$$

and $(2.5)_3$ ensures that $\mathbf{D}$ is nowhere tangent to the surface $\mathcal{S}$. If the reference configuration of $\mathcal{C}$ is specified to be the initial configuration, say at time $t = 0$, then the vector-valued functions on the right-hand sides of $(2.5)_{1,2}$ can be identified with $\mathbf{r}(\theta^\alpha, 0)$ and $\mathbf{d}(\theta^\alpha, 0)$, respectively. Analogously to (2.3), we set

$$\mathbf{D}_\alpha = \mathbf{A}_\alpha \quad , \quad \mathbf{D}_3 = \mathbf{D} \tag{2.7}$$

and note that the dual of (2.4) is given by

$$\mathbf{D}_i \cdot \mathbf{D}^j = \delta_i^j \quad . \tag{2.8}$$

As in [8], we introduce the notations grad and Grad to denote the right spatial and material gradient operators, respectively, with respect to the position on the surface $\boldsymbol{s}$ in the current configuration and on the surface $\mathcal{S}$ in the reference configuration. The corresponding divergence operators will be denoted by div and Div, respectively. In particular, for a vector-valued function $\mathbf{V}(\theta^\alpha, t)$, we have

$$\begin{aligned} \text{grad}\, \mathbf{V} &= \mathbf{V}_{,\alpha} \otimes \mathbf{a}^\alpha \quad , \quad \text{div}\, \mathbf{V} = \mathbf{V}_{,\alpha} \cdot \mathbf{a}^\alpha \quad , \\ \text{Grad}\, \mathbf{V} &= \mathbf{V}_{,\alpha} \otimes \mathbf{A}^\alpha \quad , \quad \text{Div}\, \mathbf{V} = \mathbf{V}_{,\alpha} \cdot \mathbf{A}^\alpha \quad . \end{aligned} \tag{2.9}$$

We define a measure of deformation by the tensor $\mathbf{F}$, namely[8]

$$\mathbf{F} = \mathbf{d}_i \otimes \mathbf{D}^i = \text{Grad}\, \mathbf{r} + \mathbf{d}_3 \otimes \mathbf{D}^3 \quad , \tag{2.10}$$

where the symbol $\otimes$ denotes the tensor product. Keeping the notations (2.3) and (2.7) in mind, we observe that

$$\begin{aligned} \mathbf{F}\mathbf{D}_\alpha &= \mathbf{F}\mathbf{A}_\alpha = \mathbf{a}_\alpha = \mathbf{d}_\alpha \quad , \\ \mathbf{F}\mathbf{D}_3 &= \mathbf{F}\mathbf{D} = \mathbf{d} = \mathbf{d}_3 \quad . \end{aligned} \tag{2.11}$$

By the definition of the determinant of a second order tensor and the conditions $(2.1)_3$ and $(2.5)_3$, as well as the fact that for continuous motions the scalar triple products in (2.1) and (2.5) must have the same sign, we obtain

$$\det \mathbf{F} = [\mathbf{d}_1 \mathbf{d}_2 \mathbf{d}_3]/[\mathbf{D}_1 \mathbf{D}_2 \mathbf{D}_3] > 0 \quad . \tag{2.12}$$

The tensor $\mathbf{F}$, a linear operator on vectors in 3-space, is nonsingular; and there exists, therefore, the inverse deformation gradient tensor $\mathbf{F}^{-1}$ defined by

$$\mathbf{F}^{-1} = \mathbf{D}_i \otimes \mathbf{d}^i \quad . \tag{2.13}$$

The inverse operator $\mathbf{F}^{-1}$ transforms vectors in the present configuration into vectors in the reference configuration, i.e.,

---

[8] The symbol $\mathbf{F}$ in the paper of Carroll and Naghdi [8] stands for a different quantity. The term Grad $\mathbf{r}$ in $(2.10)_2$ corresponds to the deformation gradient tensor $\mathbf{F}$ in [8].

$$\underset{\sim}{F}^{-1}\underset{\sim}{d}_i = \underset{\sim}{D}_i \tag{2.14}$$

and it follows that

$$\underset{\sim}{F}^{-1}\underset{\sim}{F} = \underset{\sim}{F}\,\underset{\sim}{F}^{-1} = \underset{\sim}{I} = \underset{\sim}{d}_i \otimes \underset{\sim}{d}^i = \underset{\sim}{D}_i \otimes \underset{\sim}{D}^i \quad , \tag{2.15}$$

where $\underset{\sim}{I}$ is the unit tensor in 3-space. We also introduce here the director gradient tensor by

$$\underset{\sim}{G} = \text{Grad}\ \underset{\sim}{d} = \underset{\sim}{d}_{3,\alpha} \otimes \underset{\sim}{D}^\alpha = \underset{\sim}{d}_{,\alpha} \otimes \underset{\sim}{D}^\alpha \quad . \tag{2.16}$$

The velocity and the director velocity vectors at a point of $\mathscr{S}$ and at time t are defined by

$$\underset{\sim}{v} = \dot{\underset{\sim}{r}}(\theta^\alpha,t) \quad , \quad \underset{\sim}{w} = \dot{\underset{\sim}{d}}(\theta^\alpha,t) \quad , \tag{2.17}$$

where a superposed dot denotes the material time derivatives with respect to t holding θ^α fixed. Since $\dot{\underset{\sim}{a}}_\alpha = \underset{\sim}{v}_{,\alpha}$, we have

$$\begin{aligned} \dot{\underset{\sim}{F}} &= \dot{\underset{\sim}{d}}_i \otimes \underset{\sim}{D}^i = \dot{\underset{\sim}{d}}_\alpha \otimes \underset{\sim}{D}^\alpha + \dot{\underset{\sim}{d}}_3 \otimes \underset{\sim}{D}^3 = \underset{\sim}{v}_{,\alpha} \otimes \underset{\sim}{D}^\alpha + \underset{\sim}{w} \otimes \underset{\sim}{D}^3 \quad , \\ \dot{\underset{\sim}{G}} &= \dot{\underset{\sim}{d}}_{3,\alpha} \otimes \underset{\sim}{D}^\alpha = \underset{\sim}{w}_{,\alpha} \otimes \underset{\sim}{D}^\alpha \quad . \end{aligned} \tag{2.18}$$

Also,

$$\begin{aligned} \dot{\underset{\sim}{F}}\,\underset{\sim}{F}^{-1} &= \dot{\underset{\sim}{d}}_i \otimes \underset{\sim}{d}^i = \text{grad}\ \underset{\sim}{v} + \underset{\sim}{w} \otimes \underset{\sim}{d}^3 \quad , \\ \dot{\underset{\sim}{G}}\,\underset{\sim}{F}^{-1} &= \underset{\sim}{w}_{,\alpha} \otimes \underset{\sim}{d}^\alpha = \text{grad}\ \underset{\sim}{w} \quad . \end{aligned} \tag{2.19}$$

3. BASIC PRINCIPLES

In the development of this section, we follow the mode of derivation of the basic theory employed by Naghdi [1, Section 8]. Let $\mathcal{P}$, bounded by a closed curve $\partial\mathcal{P}$, be any part of $\mathscr{S}$ occupied by an arbitrary material region of the surface of $\mathcal{C}$ in the present configuration at time t and let

$$\underset{\sim}{\nu} = \nu^\alpha \underset{\sim}{d}_\alpha = \nu_\alpha \underset{\sim}{d}^\alpha \tag{3.1}$$

be the outward unit normal to $\partial\mathcal{P}$. We now introduce various field quantities associated with the Cosserat surface as follows: The mass density $\rho = \rho(\theta^\gamma,t)$ of the surface $\mathscr{S}$ in the present configuration; the internal mechanical action across any curve $\partial\mathcal{P}$ characterized by the contact force[9] $\underset{\sim}{n} = \underset{\sim}{n}(\theta^\alpha,t\ ;\ \underset{\sim}{\nu})$ and the contact director force $\underset{\sim}{m} = \underset{\sim}{m}(\theta^\alpha,t\ ;\ \underset{\sim}{\nu})$, each measured per unit length of a curve in the present configuration; the intrinsic (surface) director force $\underset{\sim}{k} = \underset{\sim}{k}(\theta^\alpha,t)$ per unit area of $\mathscr{S}$, which makes no contribution to the supply of moment of momentum; the assigned force $\underset{\sim}{f} = \underset{\sim}{f}(\theta^\alpha,t)$ and the assigned director force $\underset{\sim}{\ell} = \underset{\sim}{\ell}(\theta^\alpha,t)$, each per unit mass of the surface $\mathscr{S}$; and the inertia coefficients $\alpha_M = \alpha_M(\theta^\gamma)$, $(M = 1,2)$, are assumed to be independent of time. We assume that the kinetic energy of the Cosserat surface $\mathcal{C}$ per unit area of $\mathscr{S}$ in the present configuration is given by

$$T = \tfrac{1}{2}\rho(\underset{\sim}{v}\cdot\underset{\sim}{v} + 2\alpha_1\underset{\sim}{v}\cdot\underset{\sim}{w} + \alpha_2\underset{\sim}{w}\cdot\underset{\sim}{w}) \quad . \tag{3.2}$$

We further define the momentum corresponding to the velocity $\underset{\sim}{v}$ and the director momentum corresponding to the director velocity $\underset{\sim}{w}$ by

$$\frac{\partial T}{\partial \underset{\sim}{v}} = \rho(\underset{\sim}{v} + \alpha_1\underset{\sim}{w}) \quad , \quad \frac{\partial T}{\partial \underset{\sim}{w}} = \rho(\alpha_1\underset{\sim}{v} + \alpha_2\underset{\sim}{w}) \quad . \tag{3.3}$$

---

[9] The notations for the contact force $\underset{\sim}{n}$, the contact director force $\underset{\sim}{m}$, and the surface director force $\underset{\sim}{k}$ differ from Naghdi [1] and the previous papers on the subject. In fact, the vector fields $\underset{\sim}{n},\underset{\sim}{m},\underset{\sim}{k}$ of the present paper correspond, respectively to $\underset{\sim}{N},\underset{\sim}{M},\underset{\sim}{m}$ in Naghdi [1].

Also, the physical dimensions of $\rho, \underset{\sim}{n}, \underset{\sim}{f}$ are

$$\text{phys. dim. } \rho = [ML^{-2}] \ ,$$
$$\text{phys. dim. } \underset{\sim}{n} = [MT^{-2}] \ , \quad \text{phys. dim. } \underset{\sim}{f} = [LT^{-2}] \ , \tag{3.4}$$

where the symbols [L],[M] and [T] stand for the physical dimensions of length, mass and time. The dimensions of the vector fields $\underset{\sim}{m}, \underset{\sim}{\ell}$ and $\underset{\sim}{k}$ depend upon the physical dimension of[10] $\underset{\sim}{d}$. Here, we choose $\underset{\sim}{d}$ to have the dimension of length and then $\underset{\sim}{m}, \underset{\sim}{\ell}$ will have the same physical dimensions as $\underset{\sim}{n}, \underset{\sim}{f}$ in (3.2) while $\underset{\sim}{k}$ will have the physical dimension of $[ML^{-1}T^{-2}]$.

In terms of the foregoing definitions of the various field quantities and with reference to the present configuration, the conservation laws in the purely mechanical theory of the Cosserat surface $\mathcal{C}$ are[11]

$$\frac{d}{dt}\int_{\mathcal{P}} \rho \, d\sigma = 0 \ ,$$

$$\frac{d}{dt}\int_{\mathcal{P}} \rho(\underset{\sim}{v} + \alpha_1 \underset{\sim}{w}) d\sigma = \int_{\mathcal{P}} \rho \, \underset{\sim}{f} \, d\sigma + \int_{\partial\mathcal{P}} \underset{\sim}{n} \, ds \ ,$$

$$\frac{d}{dt}\int_{\mathcal{P}} \rho(\alpha_1 \underset{\sim}{v} + \alpha_2 \underset{\sim}{w}) d\sigma = \int_{\mathcal{P}} (\rho\underset{\sim}{\ell} - \underset{\sim}{k}) d\sigma + \int_{\partial\mathcal{P}} \underset{\sim}{m} \, ds \ , \tag{3.5}$$

$$\frac{d}{dt}\int_{\mathcal{P}} \rho[\underset{\sim}{r} \times (\underset{\sim}{v} + \alpha_1 \underset{\sim}{w}) + \underset{\sim}{d} \times (\alpha_1 \underset{\sim}{v} + \alpha_2 \underset{\sim}{w})] d\sigma$$
$$= \int_{\mathcal{P}} \rho(\underset{\sim}{r} \times \underset{\sim}{f} + \underset{\sim}{d} \times \underset{\sim}{\ell}) d\sigma + \int_{\partial\mathcal{P}} (\underset{\sim}{r} \times \underset{\sim}{n} + \underset{\sim}{d} \times \underset{\sim}{m}) ds \ ,$$

where $d\sigma$ is the element of area and ds is the line element in the surface $\mathcal{S}$.

The first of (3.5) is a mathematical statement of the conservation of mass, the second that of the linear momentum, the third that of the director momentum and the fourth is the conservation of moment of momentum. The basic structure of $(3.5)_1$ and $(3.5)_2$ and their forms are analogous to the corresponding conservation laws of the classical three-dimensional continuum field theory. The structures of $(3.5)_3$ and $(3.5)_4$ are less obvious, but a motivation for their forms is provided by a derivation of the basic field equations for shell-like bodies obtained from the three-dimensional equations of continuum mechanics in which the position vector $\underset{\sim}{p}$ in 3-space is approximated by an expression of the form (1.8).

It should be noted here that the conservation laws (3.5) are consistent with the invariance conditions under superposed rigid body motions, which ordinarily have wide acceptance in continuum mechanics. Moreover, as shown in [1, Section 8], the conservation laws $(3.5)_1$, $(3.5)_2$ and $(3.5)_4$ are equivalent to, and can be derived from, an appropriate conservation of energy and the invariance conditions under superposed rigid body motions. The conservation law $(3.5)_3$ for the director momentum must be postulated separately.

Returning to the conservation laws (3.5), we note that under suitable continuity assumptions the contact force $\underset{\sim}{n}$ and the contact director force $\underset{\sim}{m}$ can be expressed in the forms (for details see [1, Section 8]):

---

[10] Depending on the choice of the physical dimension of $\underset{\sim}{d}$ and with reference to $\underset{\sim}{m}, \underset{\sim}{\ell}$ and $\underset{\sim}{k}$, the terminologies of the contact director couple, the assigned director couple and the intrinsic director couple, respectively, are also used in the literature. In particular, the latter terminologies are employed in [1], where $\underset{\sim}{d}$ is taken to be dimensionless.

[11] As the integrals on the left-hand sides of $(3.5)_{2,3,4}$ allow for coupling in inertia terms, they are slightly more general than the corresponding expressions in [1]. The conservation laws (3.5) with the coefficients $\alpha_1 = 0$, $\alpha_2 = \alpha \neq 0$ reduce to those given by equations (8.17) in [1].

$$\mathbf{n} = \mathbf{N}^{\alpha}\nu_{\alpha} = \mathbf{N}^{T}\boldsymbol{\nu} \quad , \quad \mathbf{m} = \mathbf{M}^{\alpha}\nu_{\alpha} = \mathbf{M}^{T}\boldsymbol{\nu} \quad , \tag{3.6}$$

where the second order tensors $\mathbf{N},\mathbf{M}$ and the vectors $\mathbf{N}^{\alpha},\mathbf{M}^{\alpha}$ are related through

$$\begin{aligned} \mathbf{N} &= \mathbf{d}_{\alpha} \otimes \mathbf{N}^{\alpha} \quad , \quad \mathbf{N}^{\alpha} = \mathbf{N}^{T}\mathbf{d}^{\alpha} \quad , \\ \mathbf{M} &= \mathbf{d}_{\alpha} \otimes \mathbf{M}^{\alpha} \quad , \quad \mathbf{M}^{\alpha} = \mathbf{M}^{T}\mathbf{d}^{\alpha} \end{aligned} \tag{3.7}$$

and the superscript T denotes transpose. Also, for convenience, we introduce a tensor $\mathbf{K}$ through

$$\mathbf{K} = \mathbf{d}_{3} \otimes \mathbf{k} \quad , \quad \mathbf{k} = \mathbf{K}^{T}\mathbf{d}^{3} \quad . \tag{3.8}$$

With the use of (3.6) and by usual procedures, from the conservation laws (3.5) follow the local equations in either of two equivalent forms:

$$\begin{aligned} \rho a^{\frac{1}{2}} = \lambda \quad &\text{or} \quad \dot{\rho} + \rho \mathbf{a}^{\alpha} \cdot \mathbf{v}_{,\alpha} = 0 \quad , \\ \mathbf{N}^{\alpha}{}_{|\alpha} + \rho \mathbf{f} &= \rho(\dot{\mathbf{v}} + \alpha_{1}\dot{\mathbf{w}}) \quad , \\ \mathbf{M}^{\alpha}{}_{|\alpha} + \rho \boldsymbol{\ell} - \mathbf{k} &= \rho(\alpha_{1}\dot{\mathbf{v}} + \alpha_{2}\dot{\mathbf{w}}) \quad , \\ \mathbf{a}_{\alpha} \times \mathbf{N}^{\alpha} + \mathbf{d} \times \mathbf{k} + \mathbf{d}_{,\alpha} &\times \mathbf{M}^{\alpha} = \mathbf{0} \end{aligned} \tag{3.9}$$

or

$$\begin{aligned} \dot{\rho} + \rho \operatorname{div} \mathbf{v} &= 0 \quad , \\ \operatorname{div} \mathbf{N}^{T} + \rho \mathbf{f} &= \rho(\dot{\mathbf{v}} + \alpha_{1}\dot{\mathbf{w}}) \quad , \\ \operatorname{div} \mathbf{M}^{T} + \rho \boldsymbol{\ell} - \mathbf{k} &= \rho(\alpha_{1}\dot{\mathbf{v}} + \alpha_{2}\dot{\mathbf{w}}) \quad , \\ (\mathbf{N} + \mathbf{K} + \mathbf{G}\,\mathbf{F}^{-1}\,\mathbf{M}) &= (\mathbf{N} + \mathbf{K} + \mathbf{G}\,\mathbf{F}^{-1}\,\mathbf{M})^{T} \quad , \end{aligned} \tag{3.10}$$

where λ is a function of θ^{α} only, a comma denotes partial differentiation with respect to θ^{α}, a vertical line stands for covariant differentiation with respect to the metric tensor of the surface $\mathcal{S}$ and

$$a^{\frac{1}{2}} = [\mathbf{a}_{1}\mathbf{a}_{2}\mathbf{a}_{3}] \quad . \tag{3.11}$$

Also, by the definition of the right divergence of a tensor field, we have

$$\operatorname{div} \mathbf{N}^{T} = \mathbf{N}^{\alpha}{}_{|\alpha} \quad , \quad \operatorname{div} \mathbf{M}^{T} = \mathbf{M}^{\alpha}{}_{|\alpha} \quad . \tag{3.12}$$

It is interesting that the last statement in (3.10) is similar to the symmetry of the stress in the three-dimensional theory. Furthermore, it may be observed that $\mathbf{a}_{\alpha} \times \mathbf{N}^{\alpha}$, $\mathbf{d} \times \mathbf{k}$ and $\mathbf{d}_{,\alpha} \times \mathbf{M}^{\alpha}$ are, respectively, the axial vectors of $[\mathbf{N}^{T} - \mathbf{N}]$, $[\mathbf{K}^{T} - \mathbf{K}]$ and $[(\mathbf{G}\,\mathbf{F}^{-1}\mathbf{M})^{T} - \mathbf{G}\,\mathbf{F}^{-1}\mathbf{M}]$.

At this point, it is desirable to make some observations concerning the two sets of field equations (3.9) and (3.10). If one assumes an expression of the form (1.8) for the position vector of the shell-like body, then field equations of the same form as (3.9) can also be derived from the three-dimensional field equations of classical continuum mechanics by suitable integration with respect to ξ between the limits ξ_1 and ξ_2 defined in (1.6). In a derivation of this kind, one also needs to define appropriate resultants corresponding to $\mathbf{N}^{\alpha},\mathbf{k},\mathbf{M}^{\alpha}$, as well as certain integrals corresponding to the inertia coefficients.

The assigned field $\mathbf{f}$, which occurs in the equations of motion $(3.9)_3$ and $(3.10)_2$, represents the combined effect of (i) the stress vector on the major surfaces of the shell-like body denoted by $\mathbf{f}_c$ and (ii) a contribution arising from the three-dimensional body force denoted by $\mathbf{f}_b$. A parallel statement holds for the assigned field $\boldsymbol{\ell}$ in $(3.9)_4$ and $(3.10)_3$. Thus, we may write

$$\mathbf{f} = \mathbf{f}_{b} + \mathbf{f}_{c} \quad , \quad \boldsymbol{\ell} = \boldsymbol{\ell}_{b} + \boldsymbol{\ell}_{c} \quad . \tag{3.13}$$

The various quantities in (3.13) are free to be specified in a manner which depends on the particular application in mind and, in the context of the theory of a Cosserat surface, the inertia coefficients require constitutive equations. Indeed, both $\underset{\sim}{f}_c$ and $\underset{\sim}{\ell}_c$, as well as $\underset{\sim}{f}_b$ and $\underset{\sim}{\ell}_b$, can be identified with corresponding expressions in a derivation from the three-dimensional equations (for details see [1,2]). Likewise, the inertia coefficients may be identified with easily accessible results from the three-dimensional theory.

Both sets of equations (3.9) and (3.10) are simple in appearance but they conceal the relative complexity of the results. Although it is the component forms of the equations of motion that are useful for application to specific problems, on occasions the forms (3.9) and (3.10) offer some advantages, especially in discussions pertaining to aspects of the general theory. Either of the two sets can be expressed in terms of tensor components referred to either the basis $\underset{\sim}{a}_i = (\underset{\sim}{a}_\alpha, \underset{\sim}{a}_3)$ or the basis $\underset{\sim}{d}_i = (\underset{\sim}{d}_\alpha, \underset{\sim}{d}_3)$ defined by (2.3). We do not display results of this kind here but note that the component forms of (3.9) in terms of tensor components referred to $\underset{\sim}{a}_i$ are presented and utilized in [1].

Before closing this section, we recall that the rate of work by all contact and assigned forces acting on the part $\mathcal{P}$ and on its boundary $\partial\mathcal{P}$ minus the rate of increase of the kinetic energy in $\mathcal{P}$ can be reduced to

$$\int_{\mathcal{P}} \rho(\underset{\sim}{f} \cdot \underset{\sim}{v} + \underset{\sim}{\ell} \cdot \underset{\sim}{w})d\sigma + \int_{\partial\mathcal{P}} (\underset{\sim}{n} \cdot \underset{\sim}{v} + \underset{\sim}{m} \cdot \underset{\sim}{w})ds$$
$$- \frac{d}{dt}\int_{\mathcal{P}} \frac{1}{2}\, \rho\, (\underset{\sim}{v} \cdot \underset{\sim}{v} + 2\alpha_1 \underset{\sim}{v} \cdot \underset{\sim}{w} + \alpha_2 \underset{\sim}{w} \cdot \underset{\sim}{w})d\sigma = \int_{\mathcal{P}} P\, d\sigma \quad , \tag{3.14}$$

where

$$\begin{aligned} P &= \underset{\sim}{N}^\alpha \cdot \underset{\sim}{v}_{,\alpha} + \underset{\sim}{k} \cdot \underset{\sim}{w} + \underset{\sim}{M}^\alpha \cdot \underset{\sim}{w}_{,\alpha} \\ &= \mathrm{tr}\{\underset{\sim}{N}\ \mathrm{grad}\ \underset{\sim}{v} + \underset{\sim}{K}(\underset{\sim}{w} \otimes \underset{\sim}{d}^3) + \underset{\sim}{M}\ \mathrm{grad}\ \underset{\sim}{w}\} \\ &= \mathrm{tr}\{[(\underset{\sim}{N}+\underset{\sim}{K})\dot{\underset{\sim}{F}} + \underset{\sim}{M}\,\dot{\underset{\sim}{G}}]\underset{\sim}{F}^{-1}\} \quad . \end{aligned} \tag{3.15}$$

4. ELASTIC SHELLS

Within the scope of the general theory of a Cosserat surface, we discuss briefly the constitutive equations for elastic shells in the presence of finite deformation. Preliminary to the discussion that follows, we assume the existence of a strain energy or stored energy per unit mass $\psi = \psi(\theta^\alpha, t)$ such that $\rho\dot{\psi}$ is equal to the mechanical power defined by (3.15), i.e.,

$$P = \rho\dot{\psi} \quad . \tag{4.1}$$

In passing, we note that by use of (4.1) and the definition of the strain energy for a Cosserat surface $\mathcal{C}$, namely

$$U = \int_{\mathcal{P}} \rho\, \psi\, d\sigma \quad , \tag{4.2}$$

from (3.13) follows the analogue of a well-known result in the three-dimensional theory: the rate of work by the contact and the assigned forces and director forces acting on $\mathcal{P}$ and on its boundary $\partial\mathcal{P}$ is equal to the sum of the rate of kinetic energy in $\mathcal{P}$ and the rate of the strain energy in $\mathcal{P}$.

Returning to our main objective in this section concerning the development of nonlinear constitutive equations for elastic shells, we assume that the strain energy density ψ at each material point of $\mathcal{C}$ and for all t is specified by a response function which depends on $\underset{\sim}{r}, \underset{\sim}{d}$ and their partial derivatives with respect to θ^α. But since the response function must remain unaltered under superposed rigid body translational displacement, the dependence on $\underset{\sim}{r}$ must be excluded. Thus, the constitutive assumption for the strain energy density can be written as

$$\psi = \psi'(\underset{\sim}{r}_{,\alpha}, \underset{\sim}{d}, \underset{\sim}{d}_{,\alpha} ; X) \tag{4.3}$$

and we also make similar constitutive assumptions for $\mathbf{N}^\alpha, \mathbf{k}, \mathbf{M}^\alpha$. In these constitutive equations, which represent the mechanical response of the medium, the dependence of the response functions on the local geometrical properties of a reference state and material inhomogeneity is indicated through the argument X.

A general development of various aspects of constitutive theory of elastic shells based on assumptions of the type (4.3) or variants thereof is given in [1, Section 13]. In the rest of this section, we limit the discussion to an elastic shell which is homogeneous in its reference configuration and suppose also that the dependence of the response functions on the properties of the reference state occurs through the values of the kinematical variables in the reference state. Then, in place of (4.3), we have

$$\psi = \bar{\psi}(\mathbf{r}_{,\alpha}, \mathbf{d}, \mathbf{d}_{,\alpha}; \mathbf{A}_\alpha, \mathbf{D}, \mathbf{D}_{,\alpha}) \quad , \tag{4.4}$$

with similar assumptions for $\mathbf{N}^\alpha, \mathbf{k}, \mathbf{M}^\alpha$. If instead of the kinematic variables in (4.4) the response functions are assumed to depend on Grad $\mathbf{r}$, $\mathbf{d}$, Grad $\mathbf{d}$ and their reference values, then we may write[12]

$$\psi = \bar{\bar{\psi}}(\text{Grad } \mathbf{r}, \mathbf{d}, \text{Grad } \mathbf{D} : \text{Grad } \mathbf{R}, \mathbf{D}, \text{Grad } \mathbf{D}) \quad , \tag{4.5}$$

with similar assumptions for $\mathbf{N}, \mathbf{k}, \mathbf{M}$. Still another form for the response functions could involve the tensors $\mathbf{F}, \mathbf{G}$. In this connection, we observe that the kinematic measure $\mathbf{F}$ in (2.10) already involves the reference values $\mathbf{A}_\alpha$ and $\mathbf{D}$ so that a response function of the form $\bar{\psi}$ may be expressed as a different function of $\mathbf{F}, \mathbf{G}$ and the reference value of $\mathbf{G}$. Thus, corresponding to the assumption (4.4), we may write

$$\psi = \tilde{\psi}(\mathbf{F}, \mathbf{G} : {}_R\mathbf{G}) \quad , \tag{4.6}$$

where

$${}_R\mathbf{G} = \text{Grad } \mathbf{D} = \mathbf{D}_{3,\alpha} \otimes \mathbf{D}^\alpha \quad , \tag{4.7}$$

along with similar assumptions for $\mathbf{N}, \mathbf{K}, \mathbf{M}$.

Keeping the above constitutive assumptions in mind, with the use of (4.1) and $(3.15)_{1,2,3}$, by usual techniques we obtain the following alternative forms for the constitutive equations:

$$\mathbf{N}^\alpha = \rho \frac{\partial \psi'}{\partial \mathbf{r}_{,\alpha}} \quad , \quad \mathbf{k} = \rho \frac{\partial \psi'}{\partial \mathbf{d}} \quad , \quad \mathbf{M}^\alpha = \rho \frac{\partial \psi'}{\partial \mathbf{d}_{,\alpha}} \tag{4.8}$$

or

$$\mathbf{N} = \rho \text{ Grad } \mathbf{r}\left(\frac{\partial \bar{\bar{\psi}}}{\partial \text{ Grad } \mathbf{r}}\right)^T \quad , \quad \mathbf{k} = \rho \frac{\partial \bar{\bar{\psi}}}{\partial \mathbf{d}} \quad , \quad \mathbf{M} = \rho \text{ Grad } \mathbf{r}\left(\frac{\partial \bar{\bar{\psi}}}{\partial \mathbf{G}}\right)^T \tag{4.9}$$

or

$$\mathbf{N} = \rho(\mathbf{d}_\alpha \otimes \mathbf{D}^\alpha)\left(\frac{\partial \tilde{\psi}}{\partial \mathbf{F}}\right)^T \quad , \quad \mathbf{K} = \rho(\mathbf{d}_3 \otimes \mathbf{D}^3)\left(\frac{\partial \tilde{\psi}}{\partial \mathbf{F}}\right)^T \quad , \quad \mathbf{M} = \rho \mathbf{F}\left(\frac{\partial \tilde{\psi}}{\partial \mathbf{G}}\right)^T = \rho \text{ Grad } \mathbf{r}\left(\frac{\partial \tilde{\psi}}{\partial \mathbf{G}}\right)^T . \tag{4.10}$$

In addition, the strain energy response function in (4.8) is restricted by the invariance condition[13]

$$\mathbf{r}_{,\alpha} \times \frac{\partial \psi'}{\partial \mathbf{r}_{,\alpha}} + \mathbf{d} \times \frac{\partial \psi'}{\partial \mathbf{d}} + \mathbf{d}_{,\alpha} \times \frac{\partial \psi'}{\partial \mathbf{d}_{,\alpha}} = \mathbf{0} \tag{4.11}$$

while the corresponding restriction for $\tilde{\psi}$ is

$$\mathbf{S} = \mathbf{S}^T \quad , \quad \mathbf{S} = (\mathbf{d}_\alpha \otimes \mathbf{D}^\alpha)\left(\frac{\partial \tilde{\psi}}{\partial \mathbf{F}}\right)^T + (\mathbf{d}_3 \otimes \mathbf{D}^3)\left(\frac{\partial \tilde{\psi}}{\partial \mathbf{F}}\right)^T + \mathbf{G}\left(\frac{\partial \tilde{\psi}}{\partial \mathbf{G}}\right)^T \quad . \tag{4.12}$$

---

[12] This form was employed in the paper of Carroll and Naghdi [8].

[13] Compare the combination of (4.8) and (4.11) with $(3.9)_4$.

A condition similar to $(4.12)_1$ holds for the function $\bar{\bar{\psi}}$ in (4.9).

We do not discuss here the reduced forms of the above constitutive equations resulting from invariance requirements under superposed *rigid body motions*, but for such reductions refer the reader to [1, Section 13]. Just as with the equations of motion, it is necessary in applications to specific problems to obtain alternative forms of the above constitutive equations or their reduced forms in terms of tensor components. Such component forms may be expressed with respect to bases $\mathbf{a}_i$ or $\mathbf{d}_i$, or corresponding bases in the reference configuration. Reduced forms of (4.3) have been utilized extensively in [1, Chapters D and E].

5. ADDITIONAL REMARKS

In this section we briefly comment on some special cases of the general theory and also mention some recent researches which bear on the various aspects of elastic shells. Although these developments will be described in the context of a mechanical theory, some of the references cited are somewhat more general and include thermal effects.

As noted in Section 1, the well-known membrane theory of shells can be obtained as a special case of the general theory discussed below by essentially suppressing the effect of the director and corresponding kinetic variables and this is discussed briefly in [1, Section 14]. A development of another special theory, known as the inextensional theory, wherein the length of each element of the surface of $\mathcal{C}$ is assumed to remain constant throughout all motions is also contained in [1, Section 14]. Similarly, a nonlinear restricted theory of shells by direct approach, motivated mainly by the classical theory corresponding to Kirchhoff-Love theory of shells and plates, is given by Naghdi [1, Sections 10 and 15]. A constrained theory of an elastic Cosserat surface is discussed by Green and Naghdi [10] and includes as a special case a theory which is in 1-1 correspondence with the restricted theory mentioned above.

The nonlinear constitutive equations in Section 4 are valid for an elastic Cosserat surface which may be anisotropic with reference to preferred directions associated with material points of $\mathcal{C}$. A general discussion of material symmetries for shells is given by Naghdi [1, Section 13]. Carroll and Naghdi [8] have subsequently examined the influence of the reference geometry on the response of elastic shells by assuming the existence of a local preferred state of the body and then stipulating that the influence of the reference geometry, as in (4.4), occurs through the values of the constitutive variables in the preferred state. Material symmetry restrictions for elastic shells have been discussed also, from a different point of view, by Ericksen [11] who has indicated in [12] a comparison with the results contained in [8].

Some general aspects of wave propagation in elastic shells, based on the theory of a Cosserat surface have been discussed by Ericksen [13]. A related study on the subject, limited only to wave propagation in a surface not endowed with a director, was given earlier by Cohen and Suh [14].

The theory of small deformations superposed on a large deformation of an elastic Cosserat surface, along with related problems of stability and vibrations of initially deformed plates, is discussed by Green and Naghdi [15]. For a system of linear equations characterizing the initial mixed boundary-value problem of elastic shells, Naghdi and Trapp [16] have obtained a uniqueness theorem without the use of definiteness assumptions for the strain energy density. The theorem in [16] holds for nonhomogeneous and anisotropic shells undergoing small motions superposed on a large deformation.

In still another study, the theory of a Cosserat surface has been employed by Naghdi [17] to formulate contact problems of shells and plates. In the derivation of shell theory from the three-dimensional equations, equations of motion in terms of resultants and detailed consideration of constitutive equations for shells are usually obtained relative to an interior surface, rather *than one of the major surfaces of the shell-like body which may* be the contacting surface; the interior surface ordinarily is identified with the middle surface of the shell or plate in the reference configuration. In the development of shell theory by direct approach, although the material surface of $\mathcal{C}$ may be identified with any surface of the (three-dimensional) shell-like body, nevertheless the

complete discussion of constitutive equations and the identification of the inertia coefficients and the assigned fields again require explicit use of a reference surface in the shell-like body. For certain problems it is more natural and conceptually more appealing to select one of the two major surfaces as the reference surface but then the detailed available development of the constitutive equations, as well as identification of such quantities as the inertia coefficients, have to be reconsidered relative to the new surface. This problem is resolved in [17] by deriving appropriate transformation relations which relate the kinetic variables $\underset{\sim}{N}^{\alpha}, \underset{\sim}{k}, \underset{\sim}{M}^{\alpha}$ (and hence the response functions) in the two formulations. The results in [17] are applicable to any shell-like medium and their validity is not limited to elastic shells alone.

Finally, we briefly describe here the results of a recent study by Naghdi and Tang [18] concerning controllable deformations that can be maintained, in the absence of body force, in every isotropic elastic membrane by the application of edge loads and/or uniform normal surface loads on the major surfaces of the thin shell-like body. The static solutions of finitely deformed membranes, which are valid for both compressible and incompressible materials, are obtained with the use of a strain energy response function which depends on the metric tensor of the membrane in its deformed configuration. The main results are summarized by several theorems and their corollaries in accordance with three mutually exclusive cases for which the initial undeformed surface of the membrane (which may be a sector of a complete or closed surface) is, respectively, developable, spherical and a surface of variable Gaussian curvature satisfying certain differential criteria. The corresponding deformed surfaces are, respectively, a plane or a right circular cylinder, a sphere and a surface of constant mean curvature. These results are exhaustive in that they represent all finite deformation solutions possible in every isotropic elastic material characterized by the strain energy response mentioned above. Also discussed in [18] are some special cases of the general results and several families of solutions in terms of an alternative description which should be useful in application and which permit easy interpretations.

ACKNOWLEDGMENT

The results reported here were obtained in the course of research supported by the U.S. Office of Naval Research under Contract N00014-75-C-0148, Project NR 064-436, with the University of California, Berkeley.

REFERENCES

1 Naghdi, P. M., "The Theory of Shells and Plates," in S. Flügge's Handbuch der Physik, Vol. VIa/2 (edited by C. Truesdell), Springer-Verlag, 1972, pp. 425-640.

2 Naghdi, P. M., "Direct Formulation of Some Two-Dimensional Theories of Mechanics," Proceedings of the Seventh U.S. National Congress of Applied Mechanics (Boulder, Colorado), American Society of Mechanical Engineers, 1974, pp. 3-21.

3 Duhem, P., "Le potentiel thermodynamique et la pression hydrostatique," Ann. Ecole Norm. (3), Vol. 10, 1893, pp. 187-230.

4 Cosserat, E. and F., Théorie des corps déformables, A. Hermann et Fils, Paris, 1909; also, Theory of Deformable Bodies [transl. from the original 1909 edition]. NASA TTF-11, 561 (Washington, D. C.), 1968.

5 Ericksen, J. L., and Truesdell, C., "Exact Theory of Stress and Strain in Rods and Shells," Archive for Rational Mechanics and Analysis, Vol. 1, 1958, pp. 295-323.

6 Green, A. E., Naghdi, P. M., and Wainwright, W. L., "A General Theory of a Cosserat Surface," Archive for Rational Mechanics and Analysis, Vol. 20, 1965, pp. 287-308.

7 Cohen, H., and DeSilva, C. N., "Theory of Directed Surfaces," Journal of Mathematical Physics, Vol. 7, 1966, pp. 960-966.

8 Carroll, M. M., and Naghdi, P. M., "The Influence of Reference Geometry on the Response of Elastic Shells," Archive for Rational Mechanics and Analysis, Vol. 48, 1972, pp. 302-318.

9 Green, A. E., and Naghdi, P. M., "Directed Fluid Sheets," Proceedings of the Royal Society, London, Vol. A 347, 1976, pp. 447-473.

10 Green, A. E., and Naghdi, P. M., "On the Derivation of Shell Theories by Direct Approach," Journal of Applied Mechanics, Vol. 41, 1974, pp. 173-176.

11 Ericksen, J. L., "Symmetry Transformations for Thin Elastic Shells," Archive for Rational Mechanics and Analysis, Vol. 47, 1972, pp. 1-14.

12 Ericksen, J. L., "Apparent Symmetry of Certain Thin Elastic Shells," Journal de Mécanique, Vol. 12, 1973, pp. 173-181.

13 Ericksen, J. L., "Wave Propagation in Thin Elastic Shells," Archive for Rational Mechanics and Analysis, Vol. 43, 1971, pp. 167-178.

14 Cohen, H., and Suh, S. L., "Wave Propagation in Elastic Surfaces," Journal of Mathematics and Mechanics, Vol. 19, 1970, pp. 1117-1129.

15 Green, A. E., and Naghdi, P. M., "On Superposed Small Deformations on a Large Deformation of an Elastic Cosserat Surface," Journal of Elasticity, Vol. 1, 1971, pp. 1-17.

16 Naghdi, P. M., and Trapp, J. A., "A Uniqueness Theorem in the Theory of Cosserat Surface," Journal of Elasticity, Vol. 2, 1972, pp. 9-20.

17 Naghdi, P. M., "On the Formulation of Contact Problems of Shells and Plates," Journal of Elasticity, Vol. 5, 1975, pp. 379-398.

18 Naghdi, P. M., and Tang, P. Y., "Large Deformation Possible in Every Isotropic Membrane," Philosophical Transactions of the Royal Society London, Vol. A 287, 1977, to appear.

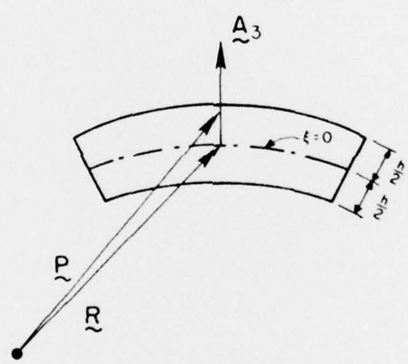


Fig. 1. An element of a shell-like body with uniform thickness in a reference configuration showing the middle surface $\xi = 0$ and the major surfaces $\xi = \pm h/2$. Also indicated are the position vector $\underset{\sim}{R}$ from a fixed origin to a point on the middle surface $\xi = 0$, the position vector $\underset{\sim}{P}$ to a point of the region of space occupied by the shell in the reference configuration and the unit normal $\underset{\sim}{A}_3$ to the middle surface.

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER (14) UCB/AM-77-7	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) (6) Shell Theory from the Standpoint of Finite Elasticity.		5. TYPE OF REPORT & PERIOD COVERED (9) Technical Report,
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) (10) P. M. Naghdi		8. CONTRACT OR GRANT NUMBER(s) (15) N00014-75-C-0148
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Mechanical Engineering University of California Berkeley, California 94720		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS NR 064-436
11. CONTROLLING OFFICE NAME AND ADDRESS Structural Mechanics Branch Office of Naval Research Arlington, Virginia 22217		12. REPORT DATE (11) September 1977
		13. NUMBER OF PAGES 13
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office) (12) 15p.		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Shell theory, nonlinear, finite deformation, basic theory, Cosserat surface, elastic shells.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This is an account of the nonlinear theory of thin shells from the standpoint of finite elasticity, together with a brief summary of some realted recent researches on the subject. The development of the basic theory, as well as a discussion of constitutive equations for elastic shells, are presented via a direct approach on the basis of a continuum model known as a Cosserat surface rather than from the three-dimensional equations of nonlinear elasticity.

DD FORM 1473 1 JAN 73 EDITION OF 1 NOV 65 IS OBSOLETE

S/N 0102-014-6601

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

400 426